

A PROGENERATOR FOR REPRESENTATIONS OF $\mathbf{SL}_n(\mathbb{F}_q)$ IN TRANSVERSE CHARACTERISTIC

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ABSTRACT. Let $G = \mathbf{GL}_n(\mathbb{F}_q)$, $\mathbf{SL}_n(\mathbb{F}_q)$ or $\mathbf{PGL}_n(\mathbb{F}_q)$, where q is a power of some prime number p , let U denote a Sylow p -subgroup of G and let R be a commutative ring in which p is invertible. Let $D(U)$ denote the derived subgroup of U and let $e = \frac{1}{|D(U)|} \sum_{u \in D(U)} u$. The aim of this note is to prove that the R -algebras RG and $eRGe$ are Morita equivalent (through the natural functor $RG\text{-mod} \rightarrow eRGe\text{-mod}$, $M \mapsto eM$).

Let n be a non-zero natural number, p a prime number, q a power of p and let \mathbb{F}_q denote a finite field with q elements. Let $G_n = \mathbf{SL}_n(\mathbb{F}_q)$. We denote by U_n the group of $n \times n$ unipotent upper triangular matrices with coefficients in \mathbb{F}_q (so that U_n is a Sylow p -subgroup of G_n). Let $D(U_n)$ denote its derived subgroup: then, with $N = (n-1)(n-2)/2$,

$$D(U_n) = \left\{ \begin{pmatrix} 1 & 0 & a_1 & \cdots & \cdots & a_{n-2} \\ 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & a_N \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \mid a_1, a_2, \dots, a_N \in \mathbb{F}_q \right\}.$$

We fix a commutative ring R in which p is invertible and we set

$$e_n = \frac{1}{|D(U_n)|} \sum_{u \in D(U_n)} u \in RD(U_n).$$

Then e_n is an idempotent of RG_n . The aim of this note is to prove the following result (recall that an idempotent i of a ring A is called *full* if $A = AiA$):

Theorem 1. *If p is invertible in R , then e_n is a full idempotent of RG_n .*

Proof. First, let $R_0 = \mathbb{Z}[1/p]$, let ζ be a primitive p -th root of unity in \mathbb{C} and let $\hat{R}_0 = R_0[\zeta]$. Let $\mathfrak{I}_0 = R_0 G_n e_n R_0 G_n$ and $\hat{\mathfrak{I}}_0 = \hat{R}_0 G_n e_n \hat{R}_0 G_n$. Since p is invertible in R , there is a unique morphism of rings $R_0 \rightarrow R$ which extends to a morphism of rings

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$R_0G_n \rightarrow RG_n$. So if $1 \in \mathfrak{I}_0$, then $1 \in \mathfrak{I}$. Also, as $(1, \zeta, \dots, \zeta^{p-2})$ is an R_0 -basis of \hat{R}_0 , it is also an R_0G_n -basis of \hat{R}_0G_n . Therefore, if $1 \in \hat{R}_0G_n e_n \hat{R}_0G_n = \hat{R}_0 \otimes_{R_0} (R_0G_n e R_0G_n)$, then $1 \in \mathfrak{I}_0$. Consequently, in order to prove Theorem 1, we may (and we shall) work under the following hypothesis:

Hypothesis. *From now on, and until the end of this proof, we assume that $R = \mathbb{Z}[1/p, \zeta]$.*

Now, let P_n denote the subgroup of $\mathbf{SL}_n(\mathbb{F}_q)$ defined by

$$P_n = \left\{ \left(\begin{array}{c|c} M & \begin{smallmatrix} a_1 \\ \vdots \\ a_{n-1} \end{smallmatrix} \\ \hline 0 & \dots & 0 & 1 \end{array} \right) \mid M \in \mathbf{SL}_{n-1}(\mathbb{F}_q) \text{ and } a_1, \dots, a_{n-1} \in \mathbb{F}_q \right\}.$$

Then $U_n \subset P_n$. We shall prove by induction on n that

(\mathcal{P}_n) e_n is a full idempotent of RP_n .

It is clear that Theorem 1 follows immediately from (\mathcal{P}_n).

As $e_1 = 1$ and $e_2 = 1$, it follows that (\mathcal{P}_1) and (\mathcal{P}_2) hold. So assume that $n \geq 3$ and that (\mathcal{P}_{n-1}) holds. Let I_n denote the identity $n \times n$ matrix and let

$$V_n = \left\{ \left(\begin{array}{c|c} I_{n-1} & \begin{smallmatrix} a_1 \\ \vdots \\ a_{n-1} \end{smallmatrix} \\ \hline 0 & \dots & 0 & 1 \end{array} \right) \mid a_1, \dots, a_{n-1} \in \mathbb{F}_q \right\}.$$

Then $V_n \simeq (\mathbb{F}_q^+)^{n-1}$ and $P_n = \mathbf{SL}_{n-1}(\mathbb{F}_q) \ltimes V_n \simeq \mathbf{SL}_{n-1}(\mathbb{F}_q) \ltimes (\mathbb{F}_q^+)^{n-1}$. We set $V'_n = D(U_n) \cap V_n$, so that $V'_n \simeq (\mathbb{F}_q^+)^{n-2}$ is normalized by P_{n-1} . Then

$$D(U_n) = D(U_{n-1}) \ltimes V'_n.$$

We now define

$$f_n = \frac{1}{|V'_n|} \sum_{v \in V'_n} v,$$

so that

$$e_n = e_{n-1} f_n.$$

By the induction hypothesis, there exists $g_1, h_1, \dots, g_l, h_l$ in P_{n-1} and r_1, \dots, r_l in R such that

$$1 = \sum_{i=1}^l r_i g_i e_{n-1} h_i.$$

Therefore, as P_{n-1} normalizes V'_n , it centralizes f_n and so

$$f_n = \left(\sum_{i=1}^l r_i g_i e_{n-1} h_i \right) f_n = \sum_{i=1}^l r_i g_i e_{n-1} f_n h_i = \sum_{i=1}^l r_i g_i e_n h_i.$$

So $f_n \in RP_n e_n RP_n$.

Let μ_p denote the subgroup of R^\times generated by ζ . If $\chi \in \text{Hom}(V_n, \mu_p)$, we define b_χ to be the associated primitive idempotent of RV_n :

$$b_\chi = \frac{1}{|V_n|} \sum_{v \in V_n} \chi(v)^{-1} v \quad \in \quad RV_n.$$

Then, as V_n is an elementary abelian p -group, we get

$$f_n = \sum_{\substack{\chi \in \text{Hom}(V_n, \mu_p) \\ \text{Res}_{V'_n}^{V_n} \chi = 1}} b_\chi.$$

We fix a non-trivial element $\chi_0 \in \text{Hom}(V_n, \mu_p)$ whose restriction to V'_n is trivial. Then

$$b_{\chi_0} = b_{\chi_0} f_n \quad \text{and} \quad b_1 = b_1 f_n,$$

so b_1 and b_{χ_0} belong to $RP_n e_n RP_n$.

But $\mathbf{SL}_{n-1}(\mathbb{F}_q) \subset P_n$ has only two orbits for its action on $\text{Hom}(V_n, \mu_p)$: the orbit of 1 and the orbit of χ_0 . Therefore, $b_\chi \in RP_n e_n RP_n$ for all $\chi \in \text{Hom}(V_n, \mu_p)$. Consequently,

$$1 = \sum_{\chi \in \text{Hom}(V_n, \mu_p)} b_\chi \in RP_n e_n RP_n,$$

as desired. \square

Finite reductive groups. Let \mathbb{F} be an algebraic closure of \mathbb{F}_q , let \mathbf{G} be a connected reductive group over \mathbb{F} and let $F : \mathbf{G} \rightarrow \mathbf{G}$ be an isogeny such that some power F^δ is a Frobenius endomorphism relative to an \mathbb{F}_q -structure. We denote by \mathbf{U} an F -stable maximal unipotent subgroup of \mathbf{G} (it is the unipotent radical of an F -stable Borel subgroup). Define

$$e = \frac{1}{|D(\mathbf{U})^F|} \sum_{u \in D(\mathbf{U})^F} u \quad \in \quad R\mathbf{G}^F.$$

The next result follows immediately from Theorem 1:

Theorem 2. *Assume that (\mathbf{G}, F) is split of type A. Then e is a full idempotent of $R\mathbf{G}^F$.*

Corollary 3. *If (\mathbf{G}, F) is split of type A, then the functors*

$$\begin{array}{ccc} R\mathbf{G}^F\text{-mod} & \longrightarrow & eR\mathbf{G}^F e\text{-mod} \\ M & \longmapsto & eM \end{array} \quad \text{and} \quad \begin{array}{ccc} R\mathbf{G}^F e\text{-mod} & \longrightarrow & R\mathbf{G}^F\text{-mod} \\ N & \longmapsto & R\mathbf{G}^F e \otimes_{eR\mathbf{G}^F e} N \end{array}$$

are mutually inverse equivalences of categories. In particular, $R\mathbf{G}^F$ and $eR\mathbf{G}^F e$ are Morita equivalent, and $R\mathbf{G}^F e$ is a progenerator for $R\mathbf{G}^F$.

Proof. This follows from Theorem 2 and, for instance, [3, Example 18.30]. \square

Examples. Theorem 2 and Corollary 3 can be applied for instance in the case where $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$, $\mathbf{SL}_n(\mathbb{F}_q)$ or $\mathbf{PGL}_n(\mathbb{F}_q)$.

Comments. (1) It is natural to ask whether Theorem 2 (or Corollary 3) can be generalized to other finite reductive groups. In fact, it cannot be generalized: indeed, if for instance $R = \mathbb{C}$, then saying that e is a full idempotent of $R\mathbf{G}^F$ means that every irreducible character of \mathbf{G}^F is an irreducible component of an Harish-Chandra induced of some Gelfand-Graev character. But, if \mathbf{G} is quasi-simple and (\mathbf{G}, F) is not split of type A , then \mathbf{G}^F contains a unipotent character which does not belong to the principal series: this character cannot be an irreducible component of an Harish-Chandra induced of a Gelfand-Graev character.

(2) In [1], a crucial step for the proof of a special case of the geometric version of Broué's abelian defect conjecture was [1, Theorem 4.1], where R. Rouquier and the author have proved the above Theorem 2 in the case where R is the integral closure of \mathbb{Z}_ℓ in a sufficiently large algebraic extension of \mathbb{Q}_ℓ (here, ℓ is a prime number different from p). The proof was essentially based on the classification, due to Dipper [2, 4.15 and 5.23], of simple modules for G_n in characteristic ℓ , and especially of cuspidal ones, which involves Deligne-Lusztig theory.

The interest of the proof given here is that it does not rely on any classification of simple modules, and is based on elementary methods: as a by-product of this elementariness, Theorem 2 and Corollary 3 are valid over any commutative ring (in which p is invertible, which is a necessary condition if one wants the idempotent e_n to be well-defined).

REFERENCES

- [1] C. BONNAFÉ & R. ROUQUIER, Coxeter orbits and modular representations, *Nagoya Math. J.* **183** (2006), 1-34.
- [2] R. DIPPER, On quotients of Hom-functors and representations of finite general linear groups II, *J. Algebra* **209** (1998), 199-269.
- [3] T.-Y. LAM, *Lectures on Modules and Rings*, Graduate Texts in Mathematics **189**, Springer, 1999, xxiv + 557 pages.

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